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NAVAL POSTGRADUATE SCHOOL

Monterey, California

THREE POSITION ESTIMATION PROCEDURES

BY

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Monterey, California

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The report describes three position estimation procedures. The first is for combining estimates from different sources into a composite position estimate. The second is for generating a position estimate from bearings on or from two or more locations. The Third is for generating a position estimate from two or more lines of bearing. The second revision represents a complete restructuring of the report. In addition to containing corrections to several errors that are present in the original version, it also contains revised versions of the two programs that they contain.

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This second revision represents a complete restructuring of the report. In addition to containing corrections of several errors that are present in the earlier versions, it also contains revised versions of the two programs that they contain. These programs are for use within the Department of the Navy and they are presented here without representation or warranty of any kind.

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I. Introduction

This report describes three procedures for estimating the position of an object. Each procedure is based on a model that describes the position of an object in terms of the position of a point on a plane surface.

The first procedure is developed in Section II. The procedure is a method for combining into a composite estimate two or more point estimates of the position of an object that each have confidence regions that are bounded by an ellipse. In the model on which it is based, the coordinates of each estimate are the values of random variables that have a bivariate normal distribution with a mean vector whose elements are the unknown coordinates of the object. A user of the procedure must specify the coordinates of each estimate and the major and minor axes of the ellipse that bounds its confidence region.

The second procedure is developed in Section III. The procedure is a method for generating a position estimate from bearings that are on or from two or more reference points of known position. In the model on which it is based, bearing errors are independent normally distributed random variables. Based on the model, the coordinates of a position estimate that is determined by the procedure are values of random variables that have a bivariate normal distribution with a mean vector whose elements are the unknown coordinates of the object. A user of the procedure must specify the standard deviations of the bearing errors. An extension of the procedure is also developed

in this section. It is a method for generating a position estimate from bearings that are on or from two or more reference points for which the position of one or more of the reference points is uncertain.

The third procedure is described in Section IV. The procedure is a method for generating a position estimate that is determined from lines of position. In the model on which the development is based, lines of position are straight lines, lines of position that are determined by observation are parallel to true lines of position and the algebraic distance between a line of position based on an observation and a true line of position is the value of a normally distributed random variable. A user of the procedure must specify the standard deviations of these random variables.

In the models on which the three procedures are based, systematic errors represent mean values that are set to zero. This suggests that the procedures should be used only were the bias in the measurements that determine bearings or lines of position is known or is known to be negligible.

II. A Composite Position Estimate

The procedure that is developed in this section is for combining position estimates for an object that are generated from data from independent sources. It is based on the following model: In a rectangular coordinate system, the coordinates of each position estimate are values of random variables that are determined by an independent bivariate normal distribution with a covariance matrix whose elements are known and with a mean vector whose elements are the unknown coordinates of the object. The above model implies that the natural logarithm of the likelihood function L for a set of n estimates can be expressed as follows: $\log L = K - \frac{1}{2} \Sigma_1^n \, (\underline{\hat{x}}_i - \underline{x}) \, \Sigma_i^{-1} \, (\underline{\hat{x}}_i - \underline{x})$ where K is a constant, $\underline{\hat{x}}_i$ is an estimate of the mean vector with components $\hat{x}_i$ and $\hat{y}_i$, $\underline{x}$ is the mean vector with components x and y, the unknown coordinates of the object, and Σ_i is the covariance matrix with elements $\sigma^2_{i\hat{x}}$, $\sigma^2_{i\hat{y}}$ and $\sigma_{i\hat{x}\hat{y}}$. The composite maximum likelihood estimates $\hat{x}$ and $\hat{y}$ of the unknown coordinates x and y make $\log L$ a maximum which implies the following relations:

$$\left.\frac{\partial(\log L)}{\partial x}\right|_{\substack{x=\hat{x}\\ y=\hat{y}}} = 0 \qquad \text{and} \qquad \left.\frac{\partial(\log L)}{\partial y}\right|_{\substack{x=\hat{x}\\ y=\hat{y}}} = 0$$

After evaluating the derivatives, they can be written as:

$$A\,\hat{x} + B\,\hat{y} = D$$

$$B\,\hat{x} + C\,\hat{y} = E$$

where:

$A = \Sigma_1^n a_i, \quad B = \Sigma_1^n b_i \;, \quad C = \Sigma_1^n c_i \;, \quad D = \Sigma_1^n (a_i\hat{x}_i + b_i\hat{y}_i) \;,$

$E = \Sigma_1^n (b_i\hat{x}_i + c_i\hat{y}_i) \;, \quad a_i = \sigma^2_{i\hat{y}} / d_i \;, \quad b_i = -\sigma_{i\hat{x}\hat{y}} / d_i \;,$

$c_i = \sigma^2_{i\hat{x}} / d_i \;, \quad d_i = \sigma^2_{i\hat{x}}\,\sigma^2_{i\hat{y}} - \sigma_{i\hat{x}\hat{y}}$

The solutions to the equations are:

$$\hat{x} = (CD - BE) / (AC - B^2)$$

$$\hat{y} = (AE - BD) / (AC - B^2)$$

Since the estimates $\hat{x}$ and $\hat{y}$ are linear combinations of the estimates $\hat{x}_i$ and $\hat{y}_i$, they are determined by a bivariate normal distribution that is defined by its mean vector with components x and y and its covariance matrix with elements $\sigma^2_{\hat{x}}$, $\sigma^2_{\hat{y}}$ and $\sigma_{\hat{x}\hat{y}}$. With $\mathcal{E}$ the expected value operator:

$$x = \mathcal{E}\{(CD - BE)/(AC - B^2)\}$$

$$y = \mathcal{E}\{(AE - BD)/(AC - B^2)\}$$

$$\sigma^2_{\hat{x}} = \mathcal{E}\{(CD - BE) - \mathcal{E}(CD - BE)\}^2/(AC - B^2)^2$$
$$= \{C^2(F+I+2L) - 2CB(H+K+M) + B^2(G+J+2N)\}/(AC - B^2)^2$$

$$\sigma^2_{\hat{y}} = \mathcal{E}\{(AE - BD) - \mathcal{E}(AE - BD)\}^2/(AC - B^2)^2$$
$$= \{A^2(G+J+2N) - 2AB(H+K+M) + B^2(F+I+2L)\}/(AC - B^2)^2$$

$$\sigma_{\hat{x}\hat{y}} = \mathcal{E}\{[(CD-BE)-\mathcal{E}(CD-BE)][(AE-BD)-\mathcal{E}(AE-BD)]\}/(AC - B^2)^2$$
$$= \{(AC+B^2)(H+K+M)-CB(F+I+2L)-BA(G+J+2N)\}/(AC - B^2)^2$$

where:

$$F = \Sigma_1^n \; a_i^2 \sigma_{i\hat{x}}^2 \;, \quad G = \Sigma_1^n \; b_i^2 \sigma_{i\hat{x}}^2 \;, \quad H = \Sigma_1^n \; a_i b_i \sigma_{i\hat{x}}^2 \;, \quad I = \Sigma_1^n \; b_i^2 \sigma_{i\hat{y}}^2 \;,$$

$$J = \Sigma_1^n \; c_i^2 \sigma_{i\hat{y}}^2 \;, \quad K = \Sigma_1^n \; b_i c_i \sigma_{i\hat{y}}^2 \;, \quad L = \Sigma_1^n \; a_i b_i \sigma_{i\hat{x}\hat{y}} \;, \quad M = \Sigma_1^n \; (a_i c_i + b_i^2) \sigma_{i\hat{x}\hat{y}}$$

$$N = \Sigma_1^n \; b_i c_i \sigma_{i\hat{x}\hat{y}}$$

Without loss of generality, the origin of the coordinate system can be the point determined by the composite estimate, its positive y-axis can be north and its positive x-axis east. In Section II of this report, it is shown that the axes of the ellipse that bounds a minimum area confidence region for a composite estimate are coincident with a primed coordinate system which is transformed to this coordinate system by a coordinate axes rotation equal to γ where γ is defined by the relation: $\tan 2\gamma = 2\sigma_{\hat{x}\hat{y}} / (\sigma_{\hat{y}}^2 - \sigma_{\hat{x}}^2)$.

It can be shown that a minimum area confidence region for a two dimensional estimator whose components have a bivariate normal distribution is bounded by an ellipse whose major axes are determined by $2k\sigma_{\hat{x}'}$ and $2k\sigma_{\hat{y}'}$ where $k = [-2 \log(1 - p)]^{1/2}$ and x' and y' refer to a coordinate system whose axes are parallel to the axes of the bounding ellipse. In this system:

$$\sigma_{\hat{x}'}^2 = \sigma_{\hat{x}}^2 \cos^2 \gamma - 2 \sigma_{\hat{x}\hat{y}} \sin \gamma \cos \gamma + \sigma_{\hat{y}}^2 \sin^2 \gamma$$

$$\sigma_{\hat{y}'}^2 = \sigma_{\hat{x}}^2 \sin^2 \gamma + 2 \sigma_{\hat{x}\hat{y}} \sin \gamma \cos \gamma + \sigma_{\hat{y}}^2 \cos^2 \gamma$$

$$\sigma_{\hat{x}'\hat{y}'} = 0$$

The model that is described above can be used to find a composite position estimate from two or more position estimates for an object. To show this, first assume that the estimates satisfy the conditions that are required by the model. Then, let δ_i be the direction of the major axis of the ellipse that bounds the confidence region that is associated with the i th position estimate. Next, orient the primed coordinate system defined above so that its x'-axis is coincident with the major axis of the ellipse. Then, the elements of the covariance matrix of the bivariate normal distribution that determines the estimate are given by the following relations:

$$\sigma^2_{i\hat{x}} = \sigma^2_{i\hat{x}'} \sin^2 \delta_i + \sigma^2_{i\hat{y}'} \cos^2 \delta_i$$

$$\sigma^2_{i\hat{y}} = \sigma^2_{i\hat{x}'} \cos^2 \delta_i + \sigma^2_{i\hat{y}'} \sin^2 \delta_i$$

$$\sigma_{i\hat{x}\hat{y}} = (\sigma^2_{i\hat{x}'} - \sigma^2_{i\hat{y}'}) \sin \delta_i \cos \delta_i$$

As an example of the use of the procedure, suppose that each bounding ellipse were a circle. In this case, the coordinates of each position estimate would be determined by a circular normal distribution and, for i = 1 to n , the elements of the covariance matrices of the circular normal distributions would satisfy the relations: $\sigma_{i\hat{x}} = \sigma_{i\hat{y}} = \sigma_i$ and $\sigma_{\hat{x}\hat{y}} = 0$. The composite estimate would be:

$$\hat{x} = [\Sigma_1^n (\hat{x}_i / \sigma_i)] / [\Sigma_1^n (1 / \sigma_i)]$$

$$\hat{y} = [\Sigma_1^n (\hat{y}_i / \sigma_i)] / [(\Sigma_1^n 1 / \sigma_i)]$$

And the elements of the covariance matrix of the distribution that determine the coordinates of the estimate would be:

$$\sigma_{\hat{x}}^2 = n \;/\; [\Sigma_1^n \;(1 \;/\; \sigma_i)]^2 \;, \quad \sigma_{\hat{y}}^2 = n \;/[(\Sigma_1^n \;(1 \;/\; \sigma_i)]^2 \quad \text{and} \quad \sigma_{\hat{x}\hat{y}} = 0 \;.$$

Consequently, the distribution would also be circular normal and the bounding curves for minimum area confidence regions would be circles.

A program called COMP that is based on the procedure that is developed in this section is listed in Appendix 1. COMP can be used to generate composite position estimates and corresponding confidence regions. As an example of its use, suppose three independent position estimates are given in Table 1.

TABLE 1

i	$\hat{x}_i$	$\hat{y}_i$	δ_i	MJ_i	MI_i	k
1	-3.7	18.1	59°	36	20	2
2	11.8	8.4	105°	38	10	2
3	0	0	146°	50	24	2

In Table 1, MJ is the length of the major axis and MI is the length of the minor axis of the ellipse bounding a minimum area confidence region. If the distance unit is the nautical mile, the coordinates of the composite estimates of the coordinates of the object are: $\hat{x} = -2.69$ nm and $\hat{y} = 12.41$ nm. For $k = 2$, the lengths of the major and minor axes of the bounding ellipse of the minimum area confidence region are: $MJ = 17.33$ nm and $MI = 8.85$ nm and the direction of its major axis is 103.77°. The results are shown in Figure 1.

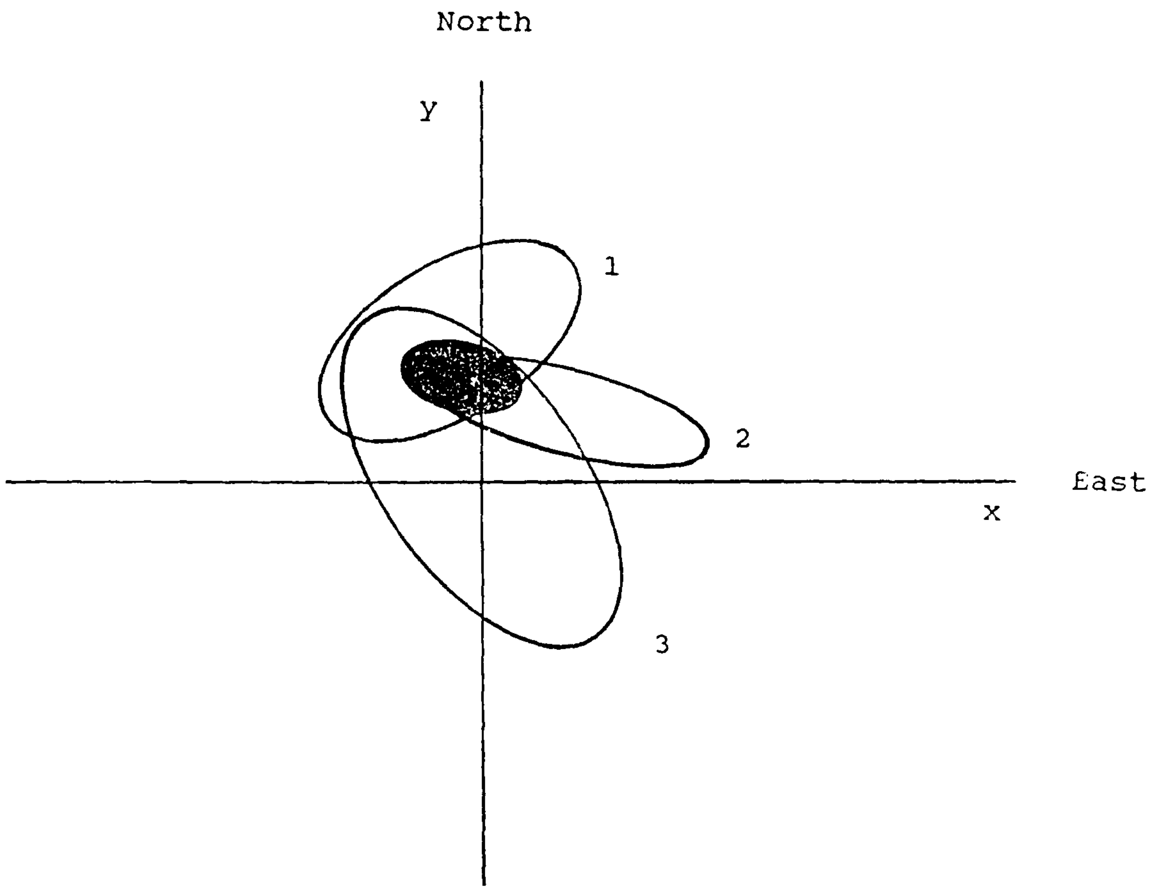

Figure 1. The position estimates and their associated confidence regions for the example on Page 7. The position estimates are at the center of their corresponding confidence region. The numbers indicate the order of the estimates in the table on Page 7. The composite confidence region is in black.

III. A Position Estimate Based on Bearings

The procedure that is developed in this section is for determining a position estimate for an object from a set of observed bearings from or on two or more reference points of known positions. In the development, reference points are referred to as stations. The procedure is based on the following model: A set of bearings taken on an object from a station are the values of independent normally distributed random variables with known standard deviations but with means that are equal to the unknown true bearings. The model implies that the natural logarithm of the likelihood function L for a set of n bearings taken on an object can be expressed as follows: $\log L = K - 1/2\ \Sigma_1^n\ (\theta_i - \phi_i)^2 / e_i^2$ where K is a constant, θ_i is an observed bearing, ϕ_i is the unknown true bearing and e_i is the known standard deviation. For $i = 1$ to n, the maximum likelihood estimates $\hat{\phi}_i$ are the solutions of the n equations:

$$\left.\frac{\partial(\log L)}{\partial \phi_i}\right|_{\phi_i=\hat{\phi}_i} = 0$$

subject to the constraint that the bearing lines determined by the estimates must all pass through a common point. Because of this constraint, the number of independent estimates is reduced from n to 2. This is equivalent to stating that any two of the estimates determine a common point. The common point is the maximum likelihood estimate for the position of the object.

In order to impose the constraint on the estimates, the likelihood function will be reformulated in a rectangular coordinate system. To do this, first, consider the following definitions:

$$u_i = r_i\ (\theta_i - \phi_i)\ ,\quad v_i = r_i\ (\phi_i - \beta_i)\quad \text{and}\quad w_i = r_i\ (\theta_i - \beta_i)$$

where $i = 1$ to n is the observation number and the observations are from two or more stations. In these definitions, β_i is the bearing of an initial estimate of the position of an object and r_i is its range from the station associated with the i th observation. It is also the radius of a circle that is centered on the station and passes through the initial estimate as shown in Figure 2. With θ_i , ϕ_i and β_i in radians, u_i , v_i and w_i are the length of arcs on this circle. And, $u_i = w_i - v_i$. In this relation, v_i involves the unknown true bearing ϕ_i , but w_i is known.

In order to express v_i in a convenient form, consider a rectangular coordinate system whose origin is at the initial estimate, whose positive y-axis is north and whose positive x-axis is east. With x and y the coordinates of the object, to first order: $v_i = x \cos \beta_i - y \sin \beta_i$. This implies:

$$\phi_i = \beta_i + (x \cos \beta_i - y \sin \beta_i) / r_i$$

And, to first order, this expresses the constraint on the ϕ_i and on their maximum likelihood estimates and it implies that the bearing lines determined by β_i , ϕ_i and θ_i are parallel.

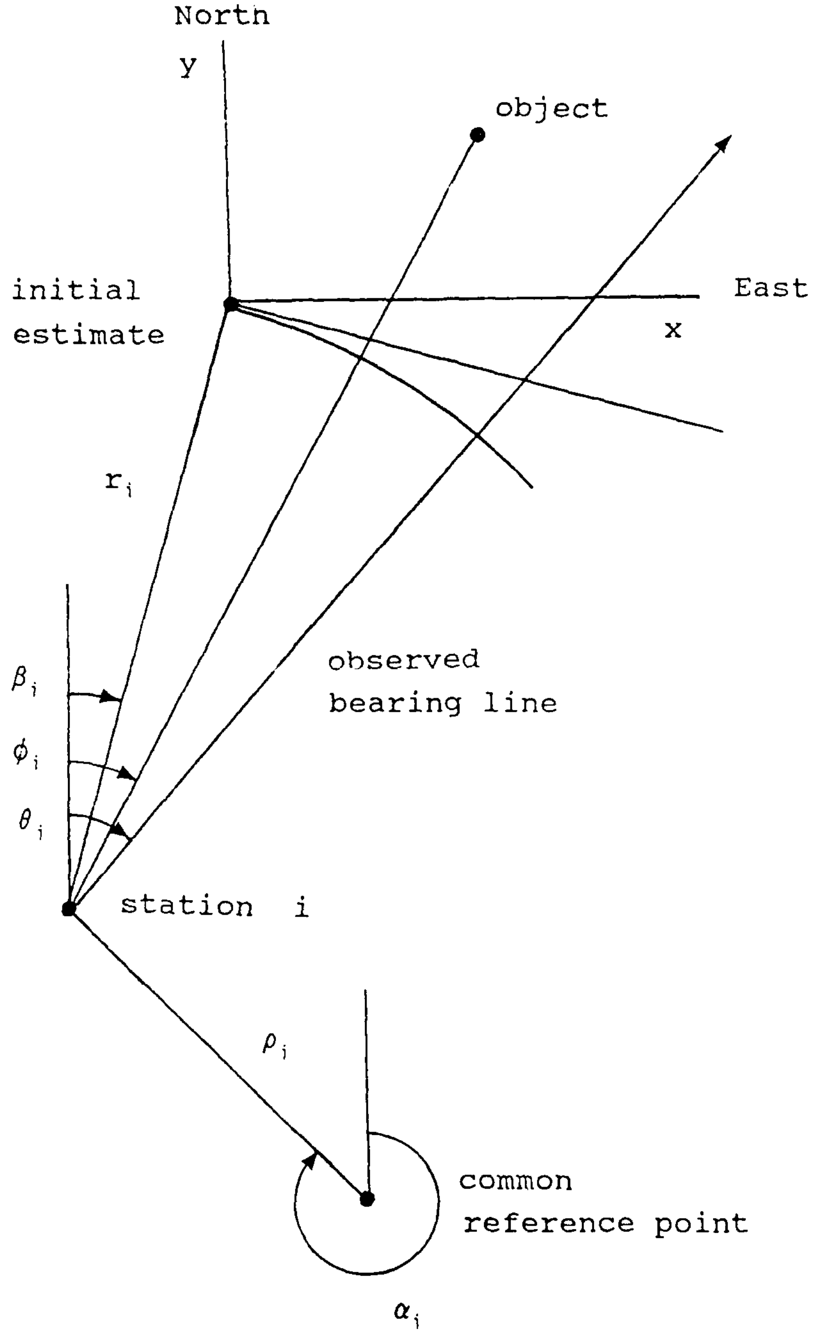

Figure 2. In the program that implements the procedure, the origin of the xy-coordinate system is coincident with the common reference point rather than with the initial estimate and the coordinates of the initial estimate are x^* and y^*. The program is discussed later in the section.

In order to determine the estimates $\hat{\phi}_i$ to first order, use the above relation to replace ϕ_i by the known values β_i and the two unknown values x and y. Next, determine the maximum likelihood estimates $\hat{x}$ and $\hat{y}$ for the unknown coordinates x and y which, to first order, are the unknown coordinates of the object. That is, find the solutions to the equations determined by:

$$\left.\frac{\partial(\log L)}{\partial x}\right|_{\substack{x=\hat{x}\\ y=\hat{y}}} = 0 \qquad \text{and} \qquad \left.\frac{\partial(\log L)}{\partial y}\right|_{\substack{x=\hat{x}\\ y=\hat{y}}} = 0$$

Evaluating the derivatives, the equations are:

$$\Sigma_1^n\ [(w_i - \hat{x}\cos\beta_i + \hat{y}\sin\beta_i)\cos\beta_i] / \sigma_i = 0$$
$$\Sigma_1^n\ [(w_i - \hat{x}\cos\beta_i + \hat{y}\sin\beta_i)\sin\beta_i] / \sigma_i = 0$$

where $\sigma_i = r_i\, e_i$. These equations can also be written as:

$$A\,\hat{x} + B\,\hat{y} = D$$
$$B\,\hat{x} + C\,\hat{y} = E$$

and their solutions as:

$$\hat{x} = (CD - BE) / (AC - B^2)$$
$$\hat{y} = (AE - BD) / (AC - B^2)$$

where

$$A = \Sigma_1^n\ (\cos^2\beta_i) / \sigma_i^2\ , \quad B = \Sigma_1^n\ (\cos\beta_i \sin\beta_i) / \sigma_i^2\ ,$$
$$C = \Sigma_1^n\ (\sin^2\beta_i) / \sigma_i^2\ , \quad D = \Sigma_1^n\ (w_i \cos\beta_i) / \sigma_i^2$$
$$E = \Sigma_1^n\ (w_i \sin\beta_i) / \sigma_i^2\ .$$

If the distribution of the random variables that determine $\hat{x}$ and $\hat{y}$ have a known distribution, then, in principle, a confidence region for an estimate can be constructed. With this in mind, the distribution of the estimators that determine $\hat{x}$ and $\hat{y}$ will be determined next. If $\hat{X}$ and $\hat{Y}$ represent the estimators, that is, the random variables that determine $\hat{x}$ and $\hat{y}$, then,

$$\hat{X} = \Sigma_1^n \; (W_i \;/\; \sigma_i^2) \; (B \sin \beta_i \; - \; C \cos \beta_i) \;/\; (B^2 - AC)$$

$$\hat{Y} = \Sigma_1^n \; (W_i \;/\; \sigma_i^2) \; (A \sin \beta_i \; - \; B \cos \beta_i) \;/\; (B^2 - AC)$$

where $W_i = r_i \; (\theta_i - \beta_i)$. Since the θ_i are normally distributed random variables, the W_i are also. And, since the estimators $\hat{X}$ and $\hat{Y}$ are linear combinations of the W_i , they have a bivariate normal distribution.

If $\beta_i = \phi_i$, for i = 1 to n, that is, if the initial estimate is the position of an object then, with E again the expected value operator, $E(W_i) = 0$, since $W_i = r_i \; (\phi_i - \beta_i)$. In this case:

$$\sigma_{\hat{x}}^2 = \Sigma_1^n \; (1/\sigma^2{}_i)(B \sin \beta_i - C \cos \beta_i)^2/(B^2 - AC)^2$$

$$\sigma_{\hat{y}}^2 = \Sigma_1^n \; (1/\sigma^2{}_i)(A \sin \beta_i - B \cos \beta_i)^2/(B^2 - AC)^2$$

$$\sigma_{\hat{x}\hat{y}} = \Sigma_1^n \; (1/\sigma^2{}_i)(B \sin \beta_i - C \cos \beta_i)(A \sin \beta_i - B \cos \beta_i)/(B^2 - AC)^2$$

Using the definitions for A , B , and C , that are given above, these equations can be written as:

$$\sigma_{\hat{x}}^2 = C / (AC - B^2)$$

$$\sigma_{\hat{y}}^2 = A / (AC - B^2)$$

$$\sigma_{\hat{x}\hat{y}} = B / (AC - B^2)$$

To the degree that the initial estimate is near the position of an object, these equations define approximations for the elements of the covariance matrix of the estimators $\hat{X}$ and $\hat{Y}$.

Since, in general, $\sigma_{\hat{x}\hat{y}}$ will not be zero, the axes of the ellipse that bounds a minimum area confidence region for $\hat{x}$ and $\hat{y}$ will not be parallel to the xy-coordinate axes. To determine the orientation, translate the xy-coordinate system so that its origin is at the point determined by $\hat{x}$ and $\hat{y}$. Then, rotate this coordinate system until its axes coincide with the axes of the bounding ellipse. Next, represent coordinates in this system by x' and y'. If the rotation angle is γ, then the coordinates of a point in the two systems are related by the following general transformation equations:

$$x' = x \cos \gamma \; - \; y \sin \gamma$$
$$y' = x \sin \gamma \; - \; y \cos \gamma$$

These transformation equations can be used to determine the elements of the covariance matrix of the distribution in the primed coordinate system from the covariance elements of the distribution matrix in an unprimed coordinate system, by using the following equations:

$$\sigma^2_{\hat{x}'} = \sigma^2_{\hat{x}} \cos^2 \gamma - 2\, \sigma_{\hat{x}\hat{y}} \sin \gamma \cos \gamma + \sigma^2_{\hat{y}} \sin^2 \gamma$$

$$\sigma^2_{\hat{y}'} = \sigma^2_{\hat{x}} \sin^2 \gamma + 2\, \sigma_{\hat{x}\hat{y}} \sin \gamma \cos \gamma + \sigma^2_{\hat{y}} \cos^2 \gamma$$

$$\sigma_{\hat{x}'\hat{y}'} = (\sigma^2_{\hat{x}} - \sigma^2_{\hat{y}}) \sin \gamma \cos \gamma + \sigma_{\hat{x}\hat{y}} (\cos^2 \gamma - \sin^2 \gamma)$$

Since $\hat{X}'$ and $\hat{Y}'$ are independent random variables when the axes of the coordinate system are parallel to the axes of the bounding ellipse, $\sigma_{\hat{x}'\hat{y}'} = 0$ which implies that γ satisfies the equation:

$$\tan 2\gamma = 2\sigma_{\hat{x}\hat{y}} / (\sigma^2_{\hat{y}} - \sigma^2_{\hat{x}}) \ .$$

If the position of an object were known, and the estimated position were at the true position, then the probability p that an estimated position would be in a region bounded by an ellipse with axes $2k\sigma_{\hat{x}'}$ and $2k\sigma_{\hat{y}'}$ that is centered on the true position is given by $p = 1 - \exp(-k^2/2)$. This result can be obtained by integrating over the region bounded by the ellipse the bivariate normal density that determines $\hat{X}'$ and $\hat{Y}'$. Note that k is a measure of the size of the region bounded by the ellipse and, for a probability of containment p, the corresponding value of k is determined by $k = [-2 \log(1 - p))]^{1/2}$. These relations provide a basis for determining confidence regions for the estimates $\hat{x}$ and $\hat{y}$ of the position of an object that are given above.

A program called PEST that is listed in Appendix 2 can be used to generate position estimates and corresponding confidence regions from bearing observations on or from two or more

stations. For a bearing on a station, θ is the reciprocal of the observed bearing. In the program, the initial estimate is the point of intersection of the two bearing lines that are determined by the first two entered bearings with each from a different station. After the first two bearings have been entered, the program determines the coordinates x^* and y^* of the initial estimate by solving the following equations:

$$x^* \sin(\theta_2 - \theta_1) = [\rho_1 \sin(\alpha_1 - \theta_1)] \sin \theta_2 - [\rho_2 \sin(\alpha_2 - \theta_2)] \sin \theta_1$$

$$y^* \sin(\theta_2 - \theta_1) = [\rho_1 \sin(\alpha_1 - \theta_1)] \cos \theta_2 - [\rho_2 \sin(\alpha_2 - \theta_2)] \cos \theta_1$$

where α_1 and ρ_1 are the bearing and range of the first station and α_2 and ρ_2 are the bearing and range of the second station from a common reference point. This method of finding the initial estimate is based on one that is used in Reference 3. The point estimates $\hat{\alpha}$ and $\hat{\rho}$ of the bearing and range of an object from a common reference point are shown in Figure 3.

As an example, consider the data in Table 2.

TABLE 2

Station Number	Observed Bearing	Station Bearing	Station Range	Bearing Error
1	38°	334°	13500	4°
2	324°	50°	11350	3°
3	3°	0°	0	4°

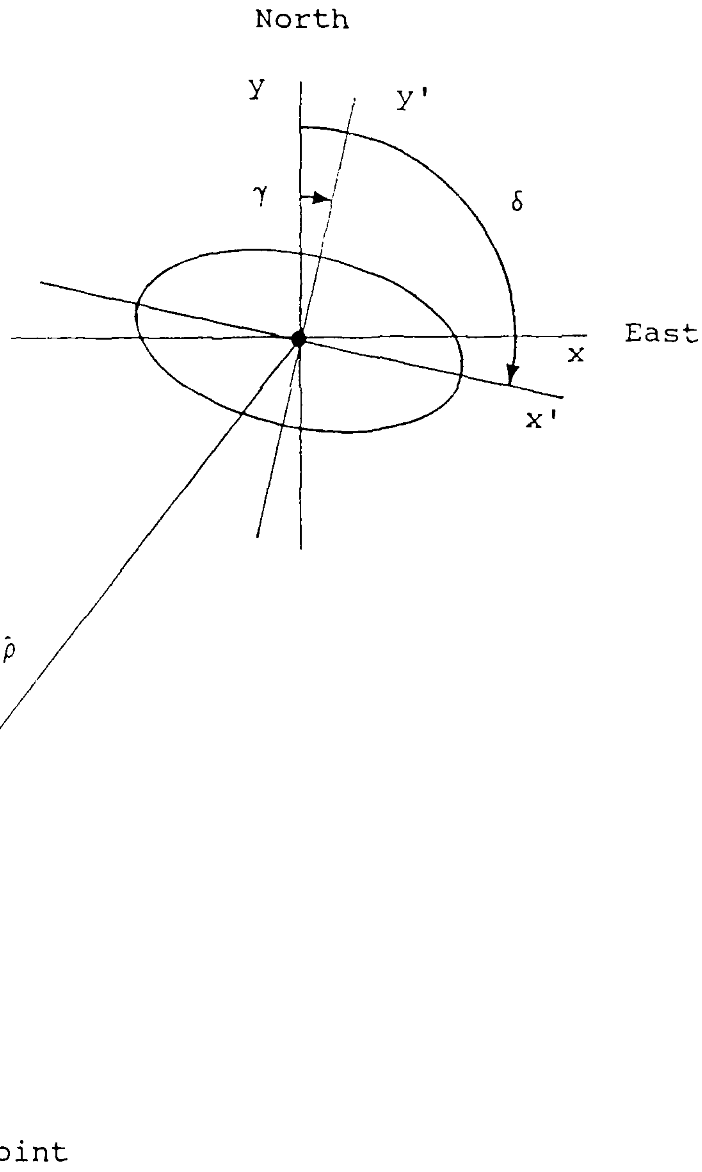

Figure 3. A confidence region bounded by an ellipse and a primed coordinate system in which $\sigma_{\hat{x}'\hat{y}'}$ is zero. The center of the ellipse and the origin of the primed and the unprimed coordinate systems are at the estimated position of an object. The direction of the major axis of the ellipse is δ where $0° \leq \delta \leq 180°$. The rotation angle from the unprimed to the primed coordinate system is γ . The estimates $\hat{\alpha}$ and $\hat{\rho}$ are the bearing and range of the object from a common reference point.

The order of data entry in the program is determined by the station number. Note that the reference point is at the location of Station 3. If the distance is in meters, then, for the three observations, the estimates generated by the program for the bearing and range of the object from Station 3 are: $\hat{\alpha} = 0.15°$ and $\hat{\rho} = 19554$ meters. For $k = 2$, the lengths of the major and minor axes of the ellipse bounding the minimum area confidence region are: MJ = 3426 meters and MI = 2260 meters and the direction of its major axis is 12.48°.

An extension of the above procedure is developed next. The extension accounts for cases where not all of the positions of the reference stations can be considered to be known. In these cases, if the station position were to become known, then, for a bearing line that is determined by an observation, the bearing line that is drawn from an assumed station position will be parallel to the bearing line that is drawn from the station position. The procedure is based on this relation and on the model with the addition of the following condition: The coordinates of an uncertain station position are determined by a distribution which is bivariate normal with a covariance matrix whose elements are known and a mean vector whose elements are the unknown coordinates of the station. This implies that for a bearing determined by an observation, the algebraic distance between the bearing line from an assumed station position and the bearing line from the station position is a random variable S that has a normal distribution with mean zero and a standard

deviation σ_s that is determined by the elements of the known covariance matrix.

The geometry of the model is shown in Figure 4. There, θ is an observed bearing and the common origin of the coordinate systems is the assumed station position. For the unprimed system, the positive y-axis direction is north and the positive x-axis direction is east. For the primed system, the x'-axis is coincident with the major axes of an ellipse that bounds a confidence region for the assumed position. The direction of the major axis is δ where $0° \leq \delta \leq 180°$. For the double primed system, the y"-axis is coincident with the assumed bearing line.

The effect of station position uncertainty is accounted for by the algebraic distance s between a bearing line from an assumed station position and the parallel bearing line from the station position. In effect, to first order, the following bearing lines are also parallel: the bearing line from a station to the initial estimate, the bearing line from the station to an object and a bearing line determined by an observation. Hence, the random variable $U_i = r_i\ (\theta_i - \phi_i)$ is the algebraic distance between the observed bearing line and the bearing line from the station to an object. And $U_{ei} = r_i\ (\theta_i - \phi_i) \pm S$ is a random variable that determines the algebraic distance between the observed bearing line from an uncertain station position and the bearing line from the station position to the object. Since U_i and S are independent random variables, the standard deviation of U_{ei} is: $\sigma_{ei} = [\sigma_i^2 + \sigma_s^2]^{1/2}$.

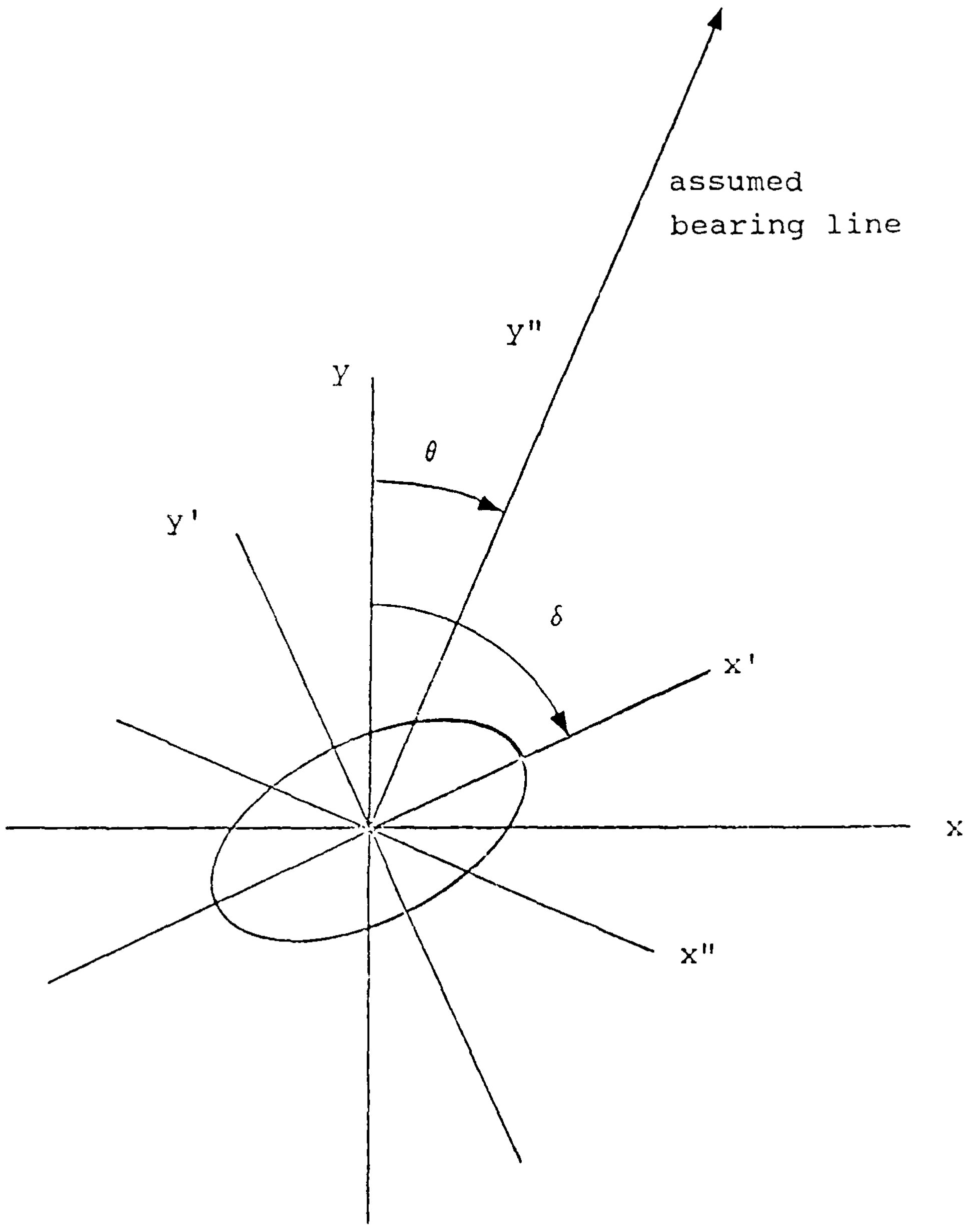

Figure 4. The geometry associated with the determination of the standard deviation σ_s . Here, θ is an observed bearing and the common origin of the coordinate systems is the assumed station position. The positive y-axis direction is north and the positive x-axis direction is east. The x'-axis is coincident with the major axes of an ellipse that bounds a confidence region for the assumed station position, δ is the direction of its major axis and the y"-axis is coincident with the assumed location of the observed bearing line.

In the extended procedure, for a station of uncertain position, the standard deviation $\sigma_i = r_i e_i$ is replaced by:

$$\sigma_{ei} = [\sigma_i^2 + \sigma_s^2]^{1/2}$$

where

$$\sigma_s^2 = \sigma_{x'}^2 \sin^2 (\delta - \theta) + \sigma_{y'}^2 \cos^2 (\delta - \theta)$$

based on Figure 4.

In Figure 4, note that $k\sigma_{x'}^2$ and $k\sigma_{y'}^2$ are the lengths of the major and minor axes of the bounding ellipse for some value of k. For an uncertain station position, β_i the bearing and r_i the range of the initial estimate from the station position are determined by the assumed station position. These relationships are illustrated in Figure 5.

In the development, although the position of one or more of the stations may be unknown, the position of a common reference point is known. This implies that the assumed station positions and the position of the initial estimate are also known.

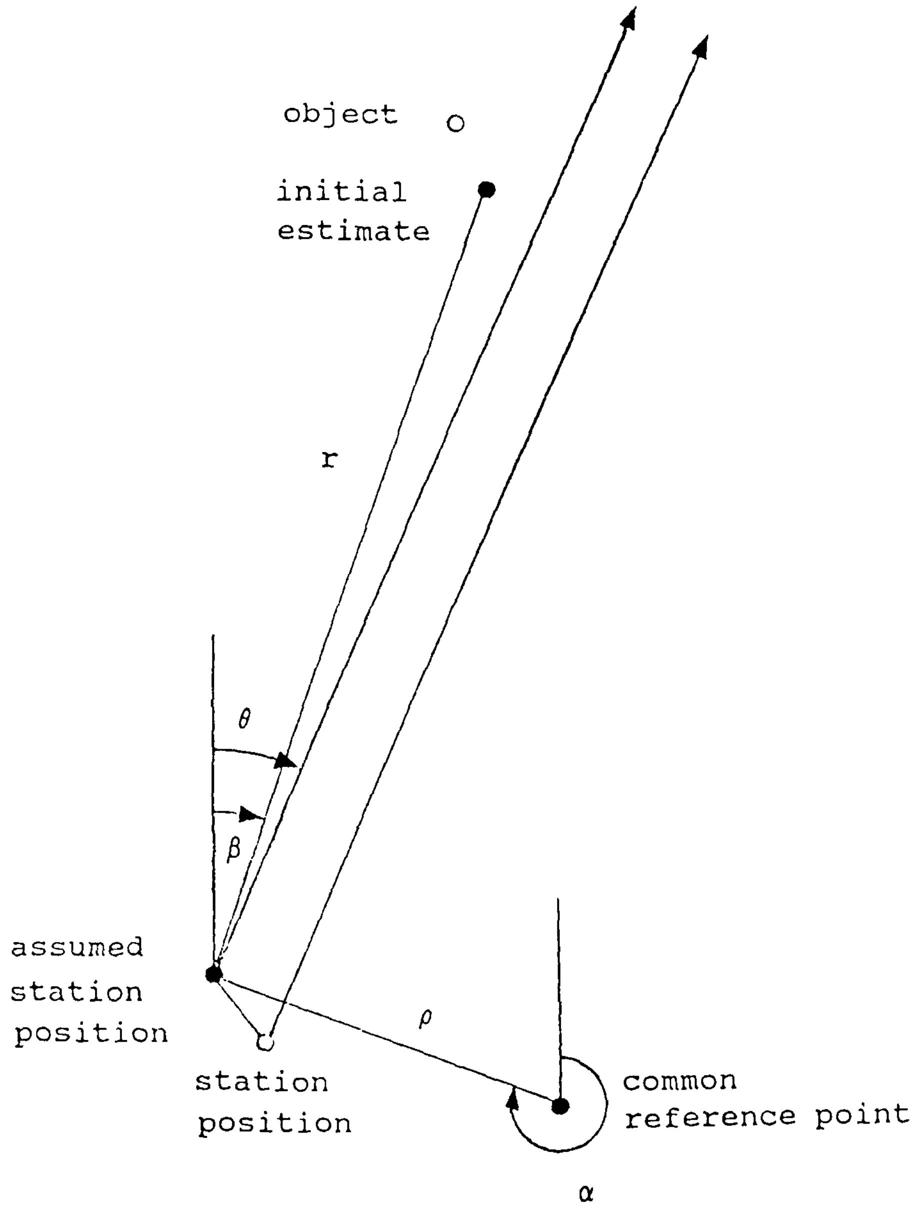

Figure 5. In the development for the extended procedure, β and r are the range and bearing of the initial estimate from the assumed station position.

IV. A Position Estimate Based on Lines of Position

The procedure that is described in this section is for determining a position estimate for an object from two or more lines of position that can be considered to be straight lines in the neighborhood of the object. The procedure is based on the following model: A line of position is a straight line. A line of position that is based on an observation is parallel to the true line of position. The algebraic distance u between a line of position that is based on an observation and the true line of position is the value of a normally distributed random variable with zero mean and known standard deviation.

Labeling lines of position by the index i, the model implies that the algebraic distance w_i between the i th line of position based on an observation and a parallel line through an initial estimate of an unknown position is related to the algebraic distance u_i between it and the true line of position by: $u_i = w_i - v_i$ where v_i is the algebraic distance between the i th true line of position and the parallel line through the initial estimate. Now, consider a rectangular coordinate system whose origin is at the initial estimate, whose positive y-axis is north and whose positive x-axis is east. With x and y the coordinates of the unknown position, to first order:

$$v_i = x \cos \beta_i - y \sin \beta_i$$

where β_i is the direction of the i th true line of position and $0° \leq \beta_i \leq 180°$. By referring to the development in Section III,

note that the estimates $\hat{x}$ and $\hat{y}$ that are developed there also apply here. However, in this development the standard deviation σ of the random variable that determines the distance u must be known for each of the observed bearing lines.

Figure 6 illustrates the geometry associated with the development.

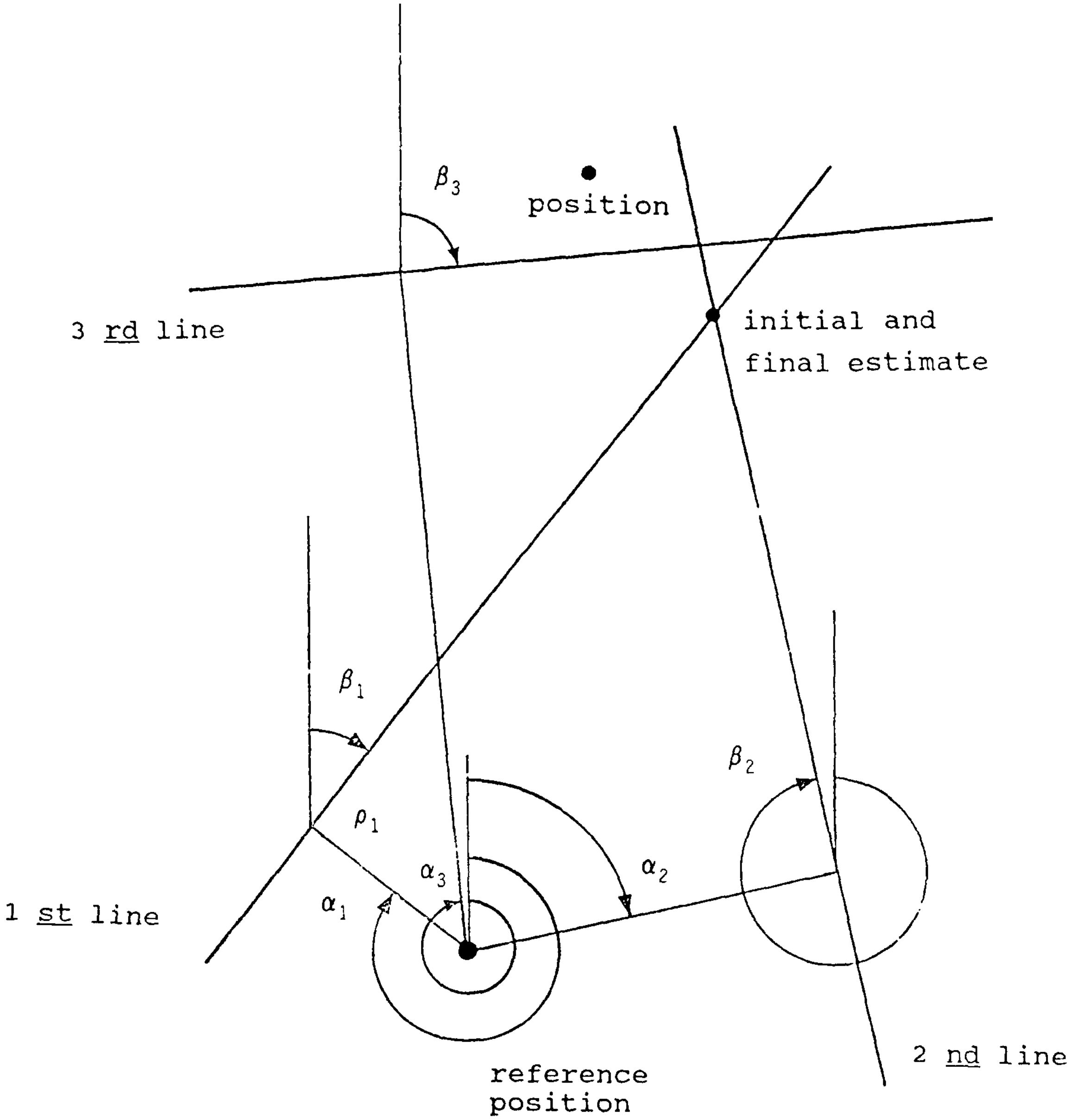

Figure 6. The geometry for three lines of position. The direction of the lines are β_1, β_2 and β_3. With the initial estimate of the unknown position at the intersection of line 1 and line 2, both w_1 and w_2 are equal to zero. This implies that the final estimate of the position and the initial estimate of the position are equal. If the three lines of position were obtained by sextant observations, the assumed position should be the reference position. In this case, the angles α_1, α_2 and α_3 would be the azimuth angles.

Appendix 1. The Program COMP

COMP is a program with which to implement the procedure developed in Section II for generating a composite estimate of the position of an object from two or more independent estimates of the position of the object. The independent estimates must be specified in terms of coordinates in a rectangular coordinate system whose positive y-axis is north and positive x-axis is east. In addition, a confidence region for each estimate must be specified in terms of the direction δ of the major axis of a bounding ellipse and either its size k or its containment probability p (confidence).

The program listing follows:

```
10 CLS : PI = 4 * ATN(1)
20 INPUT "print data yes/no (y/n)"; A$
30 IF A$ = "N" OR A$ = "n" THEN P$ = "N": GOTO 60
40 IF A$ = "Y" OR A$ = "y" THEN P$ = "Y" ELSE GOTO 20
50 LPRINT : LPRINT : LPRINT : LPRINT "Composite Position Estimation Program: COMP.BAS"
60 PRINT : INPUT "estimates to be combined"; NO
70 IF P$ = "N" THEN GOTO 90
80 LPRINT : LPRINT "estimates to be combined    ", NO
90 INPUT "elliptical confidence regions by size or probability (s/p)"; F$
100 IF F$ = "S" OR F$ = "s" OR F$ = "P" OR F$ = "p" THEN GOTO 110 ELSE GOTO 90
110 IF F$ = "P" OR F$ = "p" THEN B$ = "probability" ELSE B$ = "size"
120 FOR IO = 1 TO NO
130 PRINT : INPUT "position estimate x-coordinate"; X
140 INPUT "position estimate y-coordinate"; Y
150 INPUT "containment ellipse major axis direction"; DO
160 IF DO >= 180 OR DO < 0 THEN GOTO 150
170 INPUT "ellipse major axis length"; MA: SMA = MA / 2
180 INPUT "ellipse minor axis length"; MI: SMI = MI / 2
190 IF F$ = "P" OR F$ = "p" THEN GOTO 220
200 INPUT "containment ellipse size"; SO: PO = 1 - EXP(-SO * SO / 2)
210 PRINT "containment probability = "; PO: GOTO 240
220 INPUT "containment probability"; PO: SO = SQR(-2 * LOG(1 - PO))
230 PRINT "containment ellipse size = "; SO
240 IF P$ = "N" THEN GOTO 320
250 LPRINT : LPRINT : LPRINT "position estimate x-coordinate", X
260 LPRINT "position estimate y-coordinate", Y
270 LPRINT "ellipse major axis direction", DO
280 LPRINT "ellipse major axis length   ", MA
290 LPRINT "ellipse minor axis length   ", MI
300 LPRINT "ellipse size                ", SO
310 LPRINT "containment probability     ", PO
320 SX = SMA / SO: SY = SMI / SO
330 DO = DO * PI / 180
340 U = SX * SX * SIN(DO) * SIN(DO) + SY * SY * COS(DO) * COS(DO)
350 V = SX * SX * COS(DO) * COS(DO) + SY * SY * SIN(DO) * SIN(DO)
360 W = (SX * SX - SY * SY) * SIN(DO) * COS(DO)
370 Z = U * V - W * W: A = A + V / Z: B = B - W / Z: C = C + U / Z: D = D + V / Z * X - W / Z * Y
```

```
380 E = E - W / Z * X + U / Z * Y: F = F + V / Z * V / Z * U: G = G + W / Z * W / Z * U
390 H = H - V / Z * W / Z * U: I = I + W / Z * W / Z * V: J = J + U / Z * U / Z * V
400 K = K - W / Z * U / Z * V: L = L - V / Z * W / Z * W
410 M = M + (V / Z * U / Z + W / Z * W / Z) * W: N = N - W / Z * U / Z * W
420 NEXT IO
430 Q = A * C - B * B: X = (C * D - B * E) / Q: Y = (A * E - B * D) / Q
440 R = (C * C * (F + I + 2 * L) - 2 * C * B * (H + K + M) + B * B * (G + J + 2 * N)) / (Q * Q)
450 S = (A * A * (G + J + 2 * N) - 2 * A * B * (H + K + M) + B * B * (F + I + 2 * L)) / (Q * Q)
460 T = ((A * C + B * B) * (H + K + M) - C * B * (F + I + 2 * L) - B * A * (G + J + 2 * N)) / (Q * Q)
470 CLS : PRINT "composite estimate x-coordinate = "; X
480 PRINT "composite estimate y-coordinate = "; Y
490 IF P$ = "N" THEN GOTO 520
500 LPRINT : LPRINT : LPRINT "composite estimate x-coordinate", X
510 LPRINT "composite estimate y-coordinate", Y
520 IF S = R THEN A = 0: GOTO 600: REM  a circle
530 A = .5 * ATN(2 * T / (S - R))
540 R1 = R * COS(A) * COS(A) - 2 * T * COS(A) * SIN(A) + S * SIN(A) * SIN(A)
550 S1 = R * SIN(A) * SIN(A) + 2 * T * COS(A) * SIN(A) + S * COS(A) * COS(A)
560 A = A * 180 / PI
570 IF R1 > S1 THEN A = A + 90: GOTO 590
580 RT = R1: R1 = S1: S1 = RT
590 IF A < 0 THEN A = A + 180
600 PRINT : PRINT : INPUT "confidence region by size or probability (s/p)"; G$
610 IF G$ = "S" OR G$ = "s" THEN GOTO 630
620 IF G$ = "P" OR G$ = "p" THEN GOTO 650 ELSE GOTO 600
630 PRINT : INPUT "containment ellipse size"; S0: P0 = 1 - EXP(-S0 * S0 / 2)
640 PRINT : PRINT "containment probability = "; P0: GOTO 670
650 PRINT : INPUT "containment probability"; P0: S0 = SQR(-2 * LOG(1 - P0))
660 PRINT : PRINT "containment ellipse size = "; S0
670 PRINT "ellipse major axis length = "; 2 * S0 * SQR(R1)
680 PRINT "ellipse major axis direction = "; A
690 PRINT "ellipse minor axis length = "; 2 * S0 * SQR(S1)
700 IF P$ = "N" THEN GOTO 760
710 LPRINT : LPRINT "containment ellipse size    ", S0
720 LPRINT "containment probability       ", P0
730 LPRINT "ellipse major axis length     ", 2 * S0 * SQR(R1)
740 LPRINT "ellipse major axis direction  ", A
750 LPRINT "ellipse minor axis length     ", 2 * S0 * SQR(S1)
760 PRINT : PRINT : INPUT "continue or quit (c/q)"; G$
770 IF G$ = "C" OR G$ = "c" THEN GOTO 600
780 IF G$ = "Q" OR G$ = "q" THEN GOTO 790 ELSE GOTO 770
790 END
```

Appendix 2. The Program PEST

PEST is a program with which to implement the procedure developed in Section III for determining a position estimate for an object from a set of observed bearings from or on two or more reference points of known positions. In the development, the reference points are referred to as stations. The bearing error associated with an observation must be specified for each station. In addition, the position of each station must be specified in terms of its bearing and range from a single reference point which may be one of the reference points. The program generates an estimate for the position of an object and a confidence region for the estimate which is specified in terms of the direction δ of the major axis of a bounding ellipse and either its size k or its containment probability p (confidence).

The program listing follows:

```
10 CLS : PI = 4 * ATN(1): DIM A(7): I = 0
20 INPUT "print data (y/n)"; A$
30 IF A$ = "N" OR A$ = "n" THEN P$ = "N": GOTO 60
40 IF A$ = "Y" OR A$ = "y" THEN P$ = "Y" ELSE GOTO 20
50 LPRINT: LPRINT: LPRINT : LPRINT "Position Estimation Program: PEST.BAS"
60 INPUT "bearings on object or bearings on stations (o/s)"; B$
70 IF B$ = "O" OR B$ = "o" THEN K = 0: GOTO 120
80 IF B$ = "S" OR B$ = "s" THEN K = 1 ELSE GOTO 60
90 IF P$ = "N" THEN GOTO 120
100 IF K = 0 THEN B$ = " object" ELSE B$ = " stations"
110 LPRINT : LPRINT "bearings on" + B$: LPRINT
120 CLS : PRINT "input number " + STR$(I + 1)
130 INPUT "observed bearing in decimal degrees"; P
140 INPUT "station bearing in decimal degrees"; Q
150 INPUT "station range"; R
160 INPUT "bearing error in decimal degrees"; BE
170 IF P$ = "N" THEN GOTO 230
180 LPRINT : LPRINT "input number " + STR$(I + 1)
190 LPRINT "observed bearing in decimal degrees", P
200 LPRINT "station bearing in decimal degrees", Q
210 LPRINT "station range in chosen units", R
220 LPRINT "bearing error in decimal degrees", BE
230 IF K = 1 THEN P = P + 180
240 P = P * PI / 180: Q = Q * PI / 180: BE = BE * PI / 180
250 IF I = 2 THEN GOTO 350
260 I = I + 1: A(I - 1) = P: A(I + 1) = Q: A(I + 3) = R: A(I + 5) = BE
```

```
270 IF I = 1 THEN GOTO 120
280 X = A(4) * SIN(A(2) - A(0)): Y = A(5) * SIN(A(3) - A(1)): Z = SIN(A(1) - A(0))
290 IF Z = 0 THEN GOTO 900
300 U = (X * SIN(A(1)) - Y * SIN(A(0))) / Z: V = (X * COS(A(1)) - Y * COS(A(0))) / Z
310 FOR J = 0 TO 1
320 P = A(J): Q = A(J + 2): R = A(J + 4): BE = A(J + 6): GOSUB 940
330 NEXT J
340 GOTO 360
350 GOSUB 940
360 CLS : INPUT "continue data input or generate an estimate (c/g)"; C$
370 IF C$ = "G" OR C$ = "g" THEN GOTO 390
380 IF C$ = "C" OR C$ = "c" THEN GOTO 120 ELSE GOTO 360
390 CLS
400 F = (B * B - A * C): IF F = 0 THEN GOTO 900
410 X1 = U + (B * E - C * D) / F: Y1 = V + (A * E - B * D) / F: GOSUB 1000
420 R2 = R1: B2 = B1 * 180 / PI: IF K = 1 THEN B2 = B2 - 180: IF B2 < 0 THEN B2 = 360 + B2
430 T = SGN(B) * PI / 4: IF A = C THEN GOTO 450
440 T = .5 * ATN(2 * B / (A - C))
450 G = (C * COS(T) * COS(T) - 2 * B * COS(T) * SIN(T) + A * SIN(T) * SIN(T)) / (-F): G = SQR(G)
460 H = (C * SIN(T) * SIN(T) + 2 * B * COS(T) * SIN(T) + A * COS(T) * COS(T)) / (-F): H = SQR(H)
470 IF H >= G GOTO 490
480 Z = H: H = G: G = Z: T = T + PI / 2
490 CLS
500 PRINT "bearing estimate = "; B2
510 PRINT "range estimate = "; R2: PRINT
520 IF P$ = "N" THEN GOTO 550
530 LPRINT : LPRINT : LPRINT "bearing estimate", B2
540 LPRINT "range estimate", R2
550 INPUT "confidence region by size or probability (s/p)"; C$
560 IF C$ = "S" OR C$ = "s" THEN GOTO 580
570 IF C$ = "P" OR C$ = "p" THEN GOTO 650 ELSE GOTO 550
580 INPUT "confidence region size"; S
590 IF S <= 0 THEN GOTO 580
600 CP = 1 - EXP(-S * S / 2)
610 PRINT : PRINT "containment probability = "; CP
620 IF P$ = "N" THEN GOTO 720
630 LPRINT : LPRINT "confidence region size", S
640 LPRINT "containment probability", CP: GOTO 720
650 INPUT "containment probability (confidence)"; CP
660 IF CP >= 1 OR CP <= 0 THEN GOTO 650
670 S = SQR(-2 * LOG(1 - CP))
680 PRINT : PRINT "confidence region size = "; S
690 IF P$ = "N" THEN GOTO 720
700 LPRINT : LPRINT "containment probability", CP
710 LPRINT "confidence region size", S
720 X = 2 * S * H
730 PRINT "major axis length = "; X
740 N = T * 180 / PI: IF N < 0 THEN N = N + 180
750 PRINT "major axis direction = "; N
760 Y = 2 * S * G
770 PRINT "minor axis length = "; Y
780 PRINT "containment ellipse area = "; PI * X * Y / 4
790 IF P$ = "N" THEN GOTO 840
800 LPRINT "major axis length", X
810 LPRINT "major axis direction", N
820 LPRINT "minor axis length", Y
830 LPRINT "containment ellipse area", PI * X * Y / 4
840 PRINT : PRINT : INPUT "continue or quit (c/q)"; C$
850 IF C$ = "C" OR C$ = "c" THEN GOTO 870
860 IF C$ = "Q" OR C$ = "q" THEN END ELSE GOTO 840
870 CLS : INPUT "continue data input or generate an estimate (c/g)"; C$
880 IF C$ = "G" OR C$ = "g" THEN PRINT : GOTO 500
890 IF C$ = "C" OR C$ = "c" THEN GOTO 120 ELSE GOTO 430
900 PRINT "no solution"
910 IF P$ = "N" THEN GOTO 930
920 LPRINT : LPRINT "no solution"
930 END
940 X1 = U - R * SIN(Q): Y1 = V - R * COS(Q): GOSUB 1000
950 W = P - B1: L = R1 * BE: IF L = 0 THEN GOTO 900
```

```
960 G = COS(B1) / L: H = SIN(B1) / L
970 IF W >= PI THEN W = W - 2 * PI: GOTO 990
980 IF W <= -PI THEN W = W + 2 * PI
990 W = W / BE: A = G * G + A: B = G * H + B: C = H * H + C: D = W * G + D: E = W * H + E: RETURN
1000 R1 = SQR(X1 * X1 + Y1 * Y1): IF R1 = 0 THEN B1 = 0: RETURN
1010 IF ABS(X1 / R1) = 1 THEN M1 = SGN(X1) * PI / 2 ELSE M1 = ATN(X1 / R1 / SQR(1 - X1 * X1 / R1 / R1))
1020 IF ABS(Y1 / R1) = 1 THEN B1 = (1 - SGN(Y1)) * PI / 2 ELSE B1 = PI / 2 - ATN(Y1 / R1 / SQR(1 - Y1 * Y1 /
R1 / R1))
1030 IF M1 < 0 THEN B1 = 2 * PI - B1
1040 RETURN
```

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